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Topology Vol. 4, pp. 9–16. Pergamon Press, 1965. Printed in Great Britain

UNKNOTTING SPHERES IN CODIMENSION TWO

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(*Received* 16 *January* 1964)

§1.

IT IS A CLASSIC PROBLEM to give a homotopy theoretic criterion for an imbedding of the n-sphere S^n into a higher dimension m-sphere to be "equivalent" to the standard imbedding. To make the problem more precise, one usually chooses to work in one of three categories: differential, piecewise-linear or topological. Then, the concept of a (locally-flat) submanifold of S^m and of isomorphism (i.e. diffeomorphism, piecewise-linear homeomorphism or homeomorphism) is well-defined and the problem may be stated as follows. Let M be a submanifold of S^m, isomorphic to S^n; is there an isomorphism h of S^m such that $h(M) = S^n \subset S^m$?

Many results are known. In the differential category, if $2m > 3(n+1)$, h always exists, while if $2m \leqq 3(n+1)$, it may not [1, 18 and 16]. In the piecewise-linear and topological categories, h always exists if $m - n \geqq 3$ [11, 12, 8 and 15]. Finally, in the topological category, if $m - n = 2$ and $n \geqq 3$, h exists if and only if $S^m - M$ is homotopy equivalent to S^1 [8]. It is the main aim of this paper to examine the case $m - n = 2$ in the piecewise-linear and differential categories and show that this criterion is the correct one here also; it is necessary to exclude a few low values of n and, in the piecewise-linear situation, impose a condition of semi-local flatness.

The proofs will use the concept of spherical modifications [4, 5] and will follow almost identical lines. To avoid repetition, therefore, we will work, simultaneously, in the differential and piecewise-linear categories. Unless stated otherwise, our manifolds, submanifolds, mappings, imbeddings and isotopies will be understood to be differential or piecewise-linear, consistently. All our statements will be treated, accordingly, as referring to the differential or piecewise-linear category. All manifolds will be orientable.

§2.

Denote by D^k the unit k-disk or a k-simplex in the differential or PL category, respectively, and $S^{k-1} = \partial D^k$. Let X be a manifold and M a submanifold of X. We say M is *collared* if there is an imbedding $h : M \times D^k \to X$, where $k = \dim X - \dim M$, such that $h(x, 0) = x$, for $x \in M$. We say h is a *collar* of M.

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The main theorem of this paper will be:

THEOREM (1). *Let $n \geqq 4$; suppose M is a homology n-sphere imbedded as a closed collared submanifold of S^{n+2}. Then, if $S^{n+2} - M$ is homotopy equivalent to S^1, M bounds a contractible submanifold of S^{n+2}.*

§3.

We will need to use the notion of transverse regularity in our proof. It is, therefore, necessary to devote some attention to the *piecewise-linear* situation.

Let A be an unbounded PL-manifold, V a collared PL-submanifold. Let M be a compact PL-manifold and $f: M \to A$ a PL-map such that $f(M) \cap \partial V = \varnothing$. We say f is *transverse regular on V* if $f^{-1}(V) = N$, a collared PL-submanifold of M, with $\partial N \subset \partial M$ and codim $N =$ codim $V = k$, and there are collars h_1 of N, h_2 of V such that:

(1) $h_1 | \partial N \times \Delta^k$ is a collar of ∂N in ∂M,

(2) $fh_1(N \times \Delta^k) \subset h_2(V \times \Delta^k)$ and

(3) $h_2^{-1} fh_1 : N \times \Delta^k \to V \times \Delta^k$ is level-preserving, i.e. $\exists$ mapping $f_1 : N \times \Delta^k \to V$ so that $h_2^{-1} fh_1(x, t) = (f_1(x, t), t)$.

LEMMA (1). *Let A, V, M as above and $f: M \to A$ a PL-map which is transverse regular on V in a regular neighborhood B* [10] *of ∂M and such that $f(M) \cap \partial V = \varnothing$. Then f can be approximated by a PL-map $f' : M \to A$ which is transverse-regular on V and such that $f'|B = f|B$.*

Proof. First consider the case $A = R^k$, $V = 0$. A PL-map $f: M \to R^k$ is transverse regular at 0 if and only if there exist admissible triangulations K of M and L of R^k such that 0 is an interior point of a k-simplex of L and f is simplicial with respect to K and L. That the existence of such K and L imply transverse regularity we leave to the reader, or see [14]. Suppose f is transverse regular and $h_1 : N \times \Delta^k \to M$, $h_2 : \Delta^k \to R^k$ are collars as described above. Choose admissible triangulations K_1 of M, L_1 of R^k such that f is simplicial with respect to K_1 and L_1 and the h_i are simplicial with respect to K_1 or L_1 and admissible triangulations of $N \times \Delta^k$ or Δ^k. Let Δ_1^k be a rectilinear k-simplex such that $0 \in \operatorname{int} \Delta_1^k \subset \Delta_1^k \subset \operatorname{int} \Delta^k$; let N_1 be an admissible triangulation of N. Now define K and L as follows. On $\overline{M - h_1(N \times \Delta^k)}$ let $K = K_1$; on $\overline{R^k - h_2(\Delta^k)}$, let $L = L_1$. On $h_1(N \times \Delta_1^k)$, we carry over the product complex $N_1 \times \Delta_1^k$; on $h_2(\Delta_1^k)$, we carry over Δ_1^k. Note that f is simplicial with respect to these partial triangulations; it is an easy exercise to extend these triangulations over M and R^k so that f is still simplicial. Note that we can choose L as fine as desired.

Now consider triangulations K_1 of B and L of R^k such that 0 is an interior point of a k-simplex of L and $f|B$ is simplicial with respect to K_1 and L. We can extend K_1 to a triangulation K of M. A theorem of [13] provides a simplicial map $f' : K \to L$ such that $f'|K_1 = f|K_1$ and, by choosing L fine enough, f' may be made to approximate f as close as desired.

Now consider the general case. Let g_2 be a collar of V in A such that $g_2(\partial V \times \Delta^k) \cap f(M) = \varnothing$. Now $f^{-1}(V) \subset \operatorname{int} f^{-1}g_2(\Delta^k \times V)$; let E be a regular neighborhood of $f^{-1}(V)$ in $f^{-1}g_2(\Delta^k \times V)$. Consider $g_2^{-1}f = g' : E \to \Delta^k \times V$ and let $p : \Delta^k \times V \to \Delta^k$, $p' : \Delta^k \times V \to V$ be the projections. Now $pg' : E \to \Delta^k$ is transverse regular at 0 on a regular neighborhood B' of ∂E, where $E \cap B \subset B'$; let $\varphi : E \to \Delta^k$ be a transverse regular approximation to pg', which coincides with pg' on B'. Thus we have in E a collar g_1 of $N = \varphi^{-1}(0)$, a PL-submanifold of M, and a collar ψ of 0 in Δ^k such that $\varphi g_1(t, x) = \psi(t)$ for $t \in D^k$. Now define $g : E \to \Delta^k \times V$ by $g(x) = (\varphi(x), p'g'(x))$. Clearly $g = g'$ on B' and $pgg_1(t, x) = \psi(t)$. Now define $f' : M \to A$ by $f'|E = g_2 g$, $f'|\overline{M - E} = f|\overline{M - E}$. To exhibit the transverse regularity of f', we define collars h_1, h_2 of N, V by $h_1 = g_1$ and $h_2(t, x) = g_2(\psi t, x)$.

We remark that the differential version of Lemma (1) is proved in [9].

§4.

LEMMA (2). *Let M be a closed collared n-submanifold of S^{n+2}. Then, if $H_1(M) = 0$, M bounds a collared submanifold of S^{n+2}.*

Proof. Let g be a collar of M and $X = \overline{S^{n+2} - g(D^2 \times M)}$. Let $f : \partial X \to S^1$ be defined by projection on the "fiber". Note that f is transverse regular at every point of S^1. The only obstruction to extending f over X is in $H^2(X, \partial X; Z) \approx H_1(M; Z) = 0$. If we choose $p \in S^1$, it follows from Lemma (1) or [9] that we may choose an extension f' which is transverse regular at p. Now $f'^{-1}(p)$ is a collared submanifold of S^{n+2} with boundary $f^{-1}(p)$. We can easily alter this to obtain the desired submanifold.

Let V be a compact manifold and $\lambda : V \to I$ a mapping satisfying $\lambda^{-1}(0) = \partial V$. Define $W \subset V \times R$ as the set of points (x, t) satisfying $|t| \leqq \lambda(x)$; then W is a submanifold of $V \times R$ (with a "corner" at $\partial V \times 0$, in the differential case). Note that $\partial W = V_0 \cup V_1$, where V_t consists of the points $(x, (2t - 1)\lambda(x))$ and $V_0 \cap V_1 = \partial V_0 = \partial V_1 = \partial V \times 0$. Let $\varphi_t : V \to V_t$ be the isomorphism defined by $\varphi_t(x) = (x, (2t - 1)\lambda(x))$.

If V is a collared submanifold of an unbounded manifold A, of codimension one, then there is an imbedding $i : W \to A$ satisfying $i(x, 0) = x$ for $x \in V$. Let $Y = \overline{A - i(W)}$; then Y is homotopy equivalent to $A - V$ and $\partial Y = i(\partial W)$. Define $i_t = i\varphi_t : V \to Y$.

Suppose $f : S^k \times D^{n+1-k} \to \operatorname{int} V$ is an imbedding. We define $\theta_t(V, f) = W \cup D^{k+1} \times D^{n+1-k}$, where the "handle" is attached by the imbedding $\varphi_t f$ (in the differential case, the corners are rounded at $S^k \times S^{n-k}$); $\theta_t(V, f)$ is a manifold (with a corner at $\partial V \times 0$, in the differential case). Note that $\partial\theta_t(V, f) = V_s \cup V'$, where $s = t \pm 1$, V' is isomorphic to $\chi(V, f)$, in the notation of [4] (extended, in the natural way, to the PL case), and $\partial V' = \partial V_s = \partial V \times 0$.

For the remainder of this paper, we assume $A = S^{n+2}$, V, a collared submanifold of S^{n+2}, has dimension $n + 1$ and is $(k - 1)$-connected where $k \geqq 1$, and ∂V is homology $(k - 2)$-connected. By Alexander duality and the van Kampen and Hurewicz theorems, it follows that Y is $(k - 1)$-connected and, if $k \geqq 2$ and $H_{k-1}(\partial V) = 0$, $\pi_k(Y) \approx \pi_k(V)$.

LEMMA (3). *Suppose* (i) $n \geqq 2k+1$ *or* (ii) $n = 2k$ *or* $2k-1$ *and* $n \geqq 4$. *Then, if* $\alpha \in \pi_k(V)$ *and* $i_{t*}(\alpha) = 0$, *i can be extended to an imbedding* $i' : \theta_t(V, f) \to S^{n+2}$ *where* $f: S^k \times D^{n+1-k} \to$ *int V is an imbedding representing* α.

Remark. Clearly $i'(V')$ will be collared in A and $i'(\partial V') = \partial V$.

Proof. We begin by constructing an imbedding $g' : D^{k+1} \to Y$ such that:

$$g'(D^{k+1}) \cap \partial Y = g'(D^{k+1}) \cap V_t = g'(S^k),$$

and the intersection is normal in the differential case, and $g'|S^k = i_t f'$, where $f' : S^k \to \text{int } V$ represents α.

In the *PL* situation we can apply the results of [3] to first construct f' and then extend $i_t f'$ over D^{k+1}. In the differential situation we can, in the same way, use the results of [1] but, unfortunately, this does not cover the case $n = 5$ under hypothesis (ii). Instead we use the following argument.

Let $\alpha' \in \pi_k(\partial Y)$ correspond to α under the inclusion $V_t \subset Y$ (identifying V_t with V). Since $i_{t*}(\alpha) = 0$, α' is the boundary of an element $\beta' \in \pi_{k+1}(Y, \partial Y)$. Now Y and ∂Y are 1-connected because $k \geqq 2$ in hypothesis (ii). Since $(S^{n+2}, i(W))$ is k-connected, it follows by excision that $(Y, \partial Y)$ is k-connected. We can, therefore, apply Lemma (1) of [17] to obtain an imbedding $g'' : D^{k+1} \to Y$, representing β', such that $g''(D^{k+1})$ meets ∂Y normally along $g''(S^k)$—we assume the corners at ∂V_t are straightened. But we also need that $g''(S^k) \subset V_t$ representing α. Now V_t is 1-connected and $(V, \partial V)$ is homology $(k-1)$-connected; thus it follows by excision that $(\partial Y, V_t)$ is $(k-1)$-connected. We can then apply Lemma (2) of [17] to isotopically deform $g''(S^k)$ into V_t to represent α. An application of the isotopy extension theorem to g'' yields the desired g'.

We now would like to extend g' to an imbedding $g : D^{k+1} \times D^{n+1-k} \to Y$ such that $g(D^{k+1} \times D^{n+1-k}) \cap \partial Y = g(S^k \times D^{n+1-k}) \subset i(\text{int } V_t)$ (and, in the differential case, the intersection is normal). A tubular neighborhood of $g'(D^{k+1})$ in Y will satisfy these requirements in the differential case. In the *PL* case, choose a regular neighborhood X of $g'(D^{k+1})$ in Y such that $X \cap \partial Y = \partial X \cap \partial Y \subset i(\text{int } V_t)$ and $X \cap \partial Y$ is a regular neighborhood of $g'(S^k)$ in ∂X. It follows from [12] that $(\partial X, g'(S^k))$ is isomorphic to $(\partial D^{n+2}, \partial D^{k+1})$. Therefore, by [10], $(X, X \cap \partial Y)$ is isomorphic to $(D^{k+1} \times D^{n+1-k}, S^k \times D^{n+1-k})$. We may now define f by $\varphi_t f = g|S^k \times D^{n+1-k}$.

We will say $i'(V')$ is obtained by *killing* α.

LEMMA (4). *Suppose* (i) $i_{t*} : \pi_k(V) \to \pi_k(Y)$ *is a monomorphism for* $t = 0, 1$ *and* (ii) $\pi_k(\text{int } V) \to \pi_k(A - \partial V)$ *is zero. Then* $\pi_k(V) = 0$.

Proof. Let $\alpha \in \pi_k(V)$; by (ii) there is a mapping $f: D^{k+1} \to A - \partial V$ such that $f(S^k) \subset \text{int } V$ and $f|S^k$ represents α. We may assume f is transverse regular on V; in fact, define f in a neighborhood of S^k first, using a collar of V, and then extend over D^{k+1} and apply Lemma (1) or [9]. Thus $f^{-1}(V)$ is a, not necessarily connected, k-submanifold of D^{k+1}. We will show how to remove a component of $f^{-1}(V) \cap \text{int } D^{k+1}$, whenever one exists, leaving $f|S^k$ fixed. By a sequence of such modifications of f, we will have $f^{-1}(V) = S^k$; by transverse regularity of f, this implies $i_{t*}(\alpha) = 0$, for some t. By (i), we will have $\alpha = 0$.

Suppose $f^{-1}(V) \cap \operatorname{int} D^{k+1} \neq \varnothing$; choose an innermost component M, i.e. such that there exists a connected submanifold W of int D^{k+1} such that $\partial W = M$ and $W \cap f^{-1}(V) = M$. Consider $f|M : M \to V$; we first show that $f|M$ extends over W. In fact, the only obstruction to such an extension is the primary obstruction $\beta \in H^{k+1}(W, M; \pi_k(V)) \approx \pi_k(V)$, since V is $(k-1)$-connected. Now $i_{t*}(\beta) \in \pi_k(Y)$ is the primary obstruction to extending $i_t f|M : M \to Y$, since Y is $(k-1)$-connected. Since $W \cap f^{-1}(V) = M$ and f is transverse regular on V, it is clear that $i_t f|M$ does extend over W, for some t. Thus $i_{t*}(\beta) = 0$; by (i), this implies $\beta = 0$.

Let h_1, h_2 be collars of M, V respectively, satisfying the conditions of transverse regularity (these also exist in the differential case), (2) and (3). Assume $h_1^{-1}(W) = [-1, 0] \times M$. Let $g : \frac{1}{2} \times W \to \frac{1}{2} \times V$ be an extension of $h_2^{-1} f h_1 : \frac{1}{2} \times M \to \frac{1}{2} \times V$. Define $f_0 : W_1 \to A$, where $W_1 = W \cup h_1(D^1 \times M) \subset D^{k+1}$, as follows:

$$\begin{aligned} f_0 h_1(t, x) &= f h_1(t, x) && x \in M, \tfrac{1}{2} \leqq t \leqq 1 \\ f_0 h_1(t, x) &= f h_1(\tfrac{1}{2}, x) && x \in M, 0 \leqq t \leqq \tfrac{1}{2} \\ f_0(x) &= h_2 g(\tfrac{1}{2}, x) && x \in W. \end{aligned}$$

Note that $f_0(W_1) \cap V = \varnothing$ and $f_0 = f$ on a neighborhood of ∂W_1 (f_0 will not be differentiable). It is clear that we may now define a map $f' : D^{k+1} \to A$ such that $f'|\overline{D^{k+1} - W_1} = f|\overline{D^{k+1} - W_1}$ and $f'|W_1$ approximates f_0 closely. Obviously f' is transverse regular on V and $f'^{-1}(V) = f(V) - M$.

§5.

Suppose $n \geqq 2k + 1$; we will show how to replace V by a k-connected collared submanifold of S^{n+2} whose boundary coincides with ∂V, under the assumption $\pi_k(S^{n+2} - \partial V) \approx \pi_k(S^1)$.

First we treat the case $k = 1$. Let $\{\alpha_1, \ldots, \alpha_r\}$ be a set of generators of $\pi_1(V)$ and $f_i : D^2 \to S^{n+2} - \partial V$, $i = 1, \ldots, r$, be maps, transverse regular on V, such that $f_i(S^1) \subset \operatorname{int} V$ represents α_i. The f_i exist, as in Lemma (4), since α_i is null-homotopic in $S^{n+2} - \partial V$. We define $N(\{\alpha_i\}, \{f_i\})$ to be:

$$\sum_{i=1}^{r} \text{order } \pi_0(f_i(D^2) \cap V),$$

and $N(V)$ to be the minimum of the $N(\{\alpha_i\}, \{f_i\})$ for all choices of $\{\alpha_i\}$, $\{f_i\}$. Note the following facts.

(a) $N(V) = 0$ if and only if V is 1-connected.

(b) If an innermost component of $f_1(D^2) \cap V$ is null-homotopic in V, $N(\{\alpha_i\}, \{f_i\}) > N(V)$.

To prove (b), we use the construction in the proof of Lemma (4) to replace f_1 by f_1' such that $f_1'(D^2) \cap V$ has one less component than $f_1(D^2) \cap V$.

We now show how to replace V by a new manifold V' satisfying $N(V') < N(V)$, if $N(V) > 0$. By (a), a finite sequence of such alterations will kill $\pi_1(V)$.

Choose $\{\alpha_i\}$, $\{f_i\}$ so that $N(\{\alpha_i\}, \{f_i\}) = N(V)$. Let $\alpha \in \pi_1(V)$ be represented by an innermost component of $f_1(D^2) \cap V$; then $\alpha \in \operatorname{Ker} i_{t*}$, for some t, as is pointed out in the proof of Lemma (4). We now apply Lemma (3) to kill α to obtain our new manifold V'. According to Lemma (2) of [5], $\pi_1(V')$ is a quotient of $\pi_1(V)$ by a subgroup containing α. Let $\alpha_i' \in \pi_1(V')$ correspond to α_i. If we assume that the "handle" used in the construction of V' meets none of the $f_i(D^2)$—including $i = 1$—as we may by general position, then a slight deformation of the f_i will yield f_i', for which $N(\{\alpha_i'\}, \{f_i'\})$ is defined and equal to $N(\{\alpha_i\}, \{f_i\}) = N(V)$. But an innermost component of $f_1'(D^2) \cap V'$ is null-homotopic in V'. Therefore, by (b), $N(V') < N(V)$.

Suppose $k \geqq 2$ and i_{t*} is *not* a monomorphism. Let α be a non-zero element of $\operatorname{Ker} i_{t*}$. Let V' be obtained, according to Lemma (3), by killing α. Then V' is $(k-1)$-connected and $\pi_k(V')$ is a proper quotient of $\pi_k(V)$. Since $\pi_k(V)$ is a finitely-generated *abelian* group, this procedure may be iterated only a finite number of times, after which i_{0*} and i_{1*} will both be monomorphisms. Then, by Lemma (4), V is k-connected.

The above arguments, following an application of Lemma (2), have proved:

THEOREM (2). *Let $n \geqq 2k+1$ and M be a closed collared n-dimensional submanifold of S^{n+2}, such that $H_i(M) = 0$ for $i \leqq \max\{1, k-2\}$. Then, if $\pi_i(S^{n+2} - M) \approx \pi_i(S^1)$ for $i \leqq k$, M bounds a k-connected collared submanifold of S^{n+2}.*

Remark. The converse of this theorem is easy to prove, using the appropriate covering of $S^{n+2} - M$.

§6.

To complete the proof of Theorem (1), we must show how to kill $\pi_k(V)$, when $n = 2k$ or $2k-1$. We first treat the case $n = 2k$. Recall V is $(k-1)$-connected, and we assume $k \geqq 2$.

LEMMA (5). *Suppose V' is obtained, as in Lemma (3), by killing $\alpha \in \pi_k(V)$. If α is non-zero and of finite order, the torsion subgroup of $\pi_k(V')$ is strictly smaller than that of $\pi_k(V)$.*

Remark. If k is even, this is proved, more generally, in [4, §5].

Proof. It follows from [4, Lemma (5.6)] (clearly valid in the PL case), that $\pi_k(V)/(\alpha) \approx \pi_k(V')/(\alpha')$, where (α) is the subgroup generated by α and $\alpha' \in \pi_k(V')$. To prove Lemma (5), it suffices, by an argument in [4, p. 519] to show α' has infinite order. Suppose α is of order $p \neq 0$, α' is of order p'.

Let $\theta_t(V, f)$ have a triangulation, which extends to one of S^{n+2} under i', and has, as subcomplexes, $V \times 0$, V' and disks D, D' defined by $D = D^{k+1} \times y \cup C$ and $D' = y' \times D^{n+1-k}$, where $C =$ "cylinder" between $f(S^k)$ and $f\varphi_t(S^k)$ in W and y, y' are interior points of D^{n+1-k}, D^{k+1}, respectively. Let z, z' be cycles representing α, α', respectively carried by $\partial D \subset V \times 0$ and $\partial D' \subset V'$, respectively. Then $pz = \partial c$, $p'z' = \partial c'$, where c, c' are chains carried by $V \times 0$, V', respectively. Also $z = \partial c_1$, $z' = \partial c_1'$, where c_1, c_1' are chains carried by D, D', respectively. Note that the intersection numbers $c \cdot c_1' = c_1 \cdot c' = c \cdot c' = 0$, since $V \times 0 \cap D' = D \cap V' = V \times 0 \cap V' = \varnothing$, and $c_1 \cdot c_1' = \pm 1$, since the linking number of

$S^k \times y$ and $y' \times D^{n+1-k}$ in $D^{k+1} \times D^{n+1-k}$ is ± 1. Therefore the intersection number $(c - pc_1)\cdot(c' - p'c_1') = \pm pp'$. But the intersection number of two cycles in S^{n+2} must be zero; since $p \neq 0$, $p' = 0$.

LEMMA (6). *Let $T \subset \pi_k(V)$ be the torsion subgroup. If $i_{t*}|T$ is a monomorphism, ker i_{t*} is generated by primitive elements* (see [4, p. 516] for definition).

Proof. Let $\alpha \in \ker i_{t*}$; we shall show α is a multiple of a primitive element of ker i_{t*}. Suppose $\alpha = p\alpha'$, where α' is primitive; if $\beta = i_{t*}(\alpha')$, then $p\beta = 0$. Since $i_{t*}|T$ is a monomorphism and $\pi_k(Y) \approx \pi_k(V)$, $i_{t*}|T$ is an isomorphism onto the torsion subgroup of $\pi_k(Y)$. Therefore, $\beta = i_{t*}(\gamma)$, where $p\gamma = 0$. Let $\alpha'' = \alpha' - \gamma$; clearly $i_{t*}(\alpha'') = 0$ and α'' is primitive. Since $p\alpha'' = p\alpha' = \alpha$, this completes the proof.

Suppose $i_{t*}|T$ is *not* a monomorphism, for some t. We will describe an alteration of V which results in a new $(k-1)$-connected submanifold V' of S^{n+2}, with $\partial V' = \partial V$, and satisfying:

(i) The torsion subgroup T' of $\pi_k(V')$ is strictly smaller than that of $\pi_k(V)$, and

(ii) Ker i_{t*} (on V') contains no primitive elements.

By (i), after a finite number of such alterations, we shall have $i_{t*}|T'$ is a monomorphism, for $t = 0, 1$. But, by (ii) and Lemma (6), this means $\pi_k(V') = 0$. We have only to describe the required alteration to complete the proof of Theorem (1), when n is even.

Since $i_{t*}|T$ is not a monomorphism, we may, by Lemma (3), kill a non-zero element of ker $i_{t*}|T$. By Lemma (5), this results in a manifold with strictly smaller torsion subgroup of π_k. Next we examine the primitive elements of ker i_{t*}; if there is one, we may kill it and, by [4, p. 516], this reduces the rank of π_k by one but does not alter the torsion. Thus we may kill all the primitive elements of ker i_{t*}; the resulting manifold V' clearly satisfies (i) and (ii).

Now suppose $n = 2k - 1$; then V is $(k-1)$-connected and $\pi_k(V)$ is free abelian. Let $\alpha' \in \ker i_{t*}$; then $\alpha' = p\alpha$, where α is primitive. Since $\pi_k(Y) \approx \pi_k(V)$, $\alpha \in \ker i_{t*}$. Let V' be obtained by killing α, by Lemma (3); we shall determine $\pi_k(V')$. Note that Lemma (3) tells us that α is represented by an imbedded *collared* sphere; in particular, the self-intersection number $\alpha\cdot\alpha = 0$. Since α is primitive and the intersection pairing of V is non-singular, there exists $\beta \in \pi_k(V)$ such that $\alpha\cdot\beta = 1$. Now, by an argument of [4, p. 527] and [5, p. 54], $\pi_k(V') \approx \pi_k(V)/(\alpha, \beta)$, where (α, β) is the subgroup generated by α and β.

We see that, whenever ker $i_{t*} \neq 0$, we can reduce the rank of $\pi_k(V)$. Eventually i_{t*} will be a monomorphism for $t = 0, 1$; by Lemma (4), $\pi_k(V) = 0$. This completes the proof of Theorem (1).

§7.

We conclude by proving the promised unknotting theorem.

THEOREM (3). *Let M be a homotopy n-sphere imbedded in S^{n+2} such that $S^{n+2} - M$ is homotopy equivalent to S^1. Then, if $n \geqq 5$ there is an isomorphism h of S^{n+2} onto itself such that $h(M)$ is the standard $S^n \subset S^{n+2}$. If $n = 4$, the conclusion follows if we assume that M is already isomorphic to S^n.*

Proof. By Theorem (1), M bounds a contractible submanifold V of S^{n+2}. By [2, Theorem (3.1)], we may assume V has a compatible differential structure; if $n = 4$, it follows from an unpublished result of Cerf, that M is diffeomorphic to S^4. By a result of [7], V is now diffeomorphic to D^{n+1}, since ∂V is simply-connected and, if $n = 4$, is diffeomorphic to S^4. Therefore V was already isomorphic to D^{n+1}. The theorem now follows from [6, Theorem (B)] and [11, p. 354].

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